

Mathematical Journal of Okayama University

Volume 44, Issue 1

2002

Article 10

JANUARY 2002

Decomposition of Spinor Groups by the Involution σ' in Exceptional Lie Groups

Toshikazu Miyashita*

*Tohbu High School

Copyright ©2002 by the authors. *Mathematical Journal of Okayama University* is produced by
The Berkeley Electronic Press (bepress). <http://escholarship.lib.okayama-u.ac.jp/mjou>

Math. J. Okayama Univ. 44(2002), 1–27

DECOMPOSITION OF SPINOR GROUPS BY THE INVOLUTION σ' IN EXCEPTIONAL LIE GROUPS

TOSHIKAZU MIYASHITA

INTRODUCTION

The compact exceptional Lie groups F_4 , E_6 , E_7 and E_8 have spinor groups as a subgroup as follows:

$$\begin{array}{l} F_4 \supset Spin(9) \supset Spin(8) \supset Spin(7) \supset \cdots \supset Spin(1) \ni 1 \\ \cap \\ E_6 \supset Spin(10) \\ \cap \\ E_7 \supset Spin(12) \supset Spin(11) \\ \cap \\ E_8 \supset Ss(16) \supset Spin(15) \supset Spin(14) \supset Spin(13). \end{array}$$

On the other hand, we know the involution σ' induced an element $\sigma' \in Spin(8) \subset F_4 \subset E_6 \subset E_7 \subset E_8$. Now, in this paper, we determine the group structures of $(Spin(n))^{\sigma'}$ which are the fixed subgroups by the involution σ' . Our results are as follows:

$$\begin{array}{ll} F_4 & (Spin(9))^{\sigma'} \cong Spin(8), \\ E_6 & (Spin(10))^{\sigma'} \cong (Spin(2) \times Spin(8))/\mathbf{Z}_2, \\ E_7 & (Spin(11))^{\sigma'} \cong (Spin(3) \times Spin(8))/\mathbf{Z}_2, \\ & (Spin(12))^{\sigma'} \cong (Spin(4) \times Spin(8))/\mathbf{Z}_2, \\ E_8 & (Spin(13))^{\sigma'} \cong (Spin(5) \times Spin(8))/\mathbf{Z}_2, \\ & (Spin(14))^{\sigma'} \cong (Spin(6) \times Spin(8))/\mathbf{Z}_2. \end{array}$$

Needless to say, the spinor groups appeared in the first term have relation

$$Spin(2) \subset Spin(3) \subset Spin(4) \subset Spin(5) \subset Spin(6).$$

One of our aims is to find these groups explicitly in the exceptional groups. In the group E_8 , we conjecture that

$$\begin{array}{l} (Spin(15))^{\sigma'} \cong (Spin(7) \times Spin(8))/\mathbf{Z}_2, \\ (Ss(16))^{\sigma'} \cong (Spin(8) \times Spin(8))/(\mathbf{Z}_2 \times \mathbf{Z}_2), \end{array}$$

however, we can not realize explicitly.

This paper is closely in connection with the preceding papers [2], [3], [4] and may be a continuation of [2], [3], [4] in some sense.

1. GROUP F_4

We use the same notation as in [5] (however, some will be rewritten). For example,

- the Cayley algebra $\mathfrak{C} = \mathbf{H} \oplus \mathbf{H}e_4$,
- the exceptional Jordan algebra $\mathfrak{J} = \{X \in M(3, \mathfrak{C}) \mid X^* = X\}$, the Jordan multiplication $X \circ Y$, the inner product (X, Y) and the elements $E_1, E_2, E_3 \in \mathfrak{J}$,
- the group $F_4 = \{\alpha \in \text{Iso}_{\mathbf{R}}(\mathfrak{J}) \mid \alpha(X \circ Y) = \alpha X \circ \alpha Y\}$, and the element $\sigma \in F_4$: $\sigma X = DXD$, $D = \text{diag}(1, -1, -1)$, $X \in \mathfrak{J}$ and the element $\sigma' \in F_4$: $\sigma' X = D'XD'$, $D' = \text{diag}(-1, -1, 1)$, $X \in \mathfrak{J}$,
- the groups $SO(8) = SO(\mathfrak{C})$ and $\underline{Spin}(8) = \{(\alpha_1, \alpha_2, \alpha_3) \in SO(8) \times SO(8) \times SO(8) \mid (\alpha_1 x)(\alpha_2 y) = \alpha_3(\overline{xy})\}$.

Proposition 1.1. $(F_4)_{E_1} \cong Spin(9)$.

Proof. We define a 9-dimensional \mathbf{R} -vector space V^9 by

$$V^9 = \{X \in \mathfrak{J} \mid E_1 \circ X = 0, \text{tr}(X) = 0\} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi & x \\ 0 & \bar{x} & -\xi \end{pmatrix} \mid \xi \in \mathbf{R}, x \in \mathfrak{C} \right\}$$

with the norm $1/2(X, X) = \xi^2 + \bar{x}x$. Let $SO(9) = SO(V^9)$. Then, we have $(F_4)_{E_1}/\mathbf{Z}_2 \cong SO(9)$, $\mathbf{Z}_2 = \{1, \sigma\}$. Therefore, $(F_4)_{E_1}$ is isomorphic to $Spin(9)$ as a double covering group of $SO(9)$. (In detail, see [5], [8]). \square

Now, we shall determine the group structure of $(Spin(9))^{\sigma'}$.

Theorem 1.2. $(Spin(9))^{\sigma'} \cong Spin(8)$.

Proof. Let $Spin(9) = (F_4)_{E_1}$. Then, the map $\varphi_1: Spin(8) \rightarrow (Spin(9))^{\sigma'}$,

$$\varphi_1(\alpha_1, \alpha_2, \alpha_3)X = \begin{pmatrix} \xi_1 & \alpha_3 x_3 & \overline{\alpha_2 x_2} \\ \overline{\alpha_3 x_3} & \xi_2 & \alpha_1 x_1 \\ \alpha_2 x_2 & \overline{\alpha_1 x_1} & \xi_3 \end{pmatrix}, \quad X \in \mathfrak{J}$$

gives an isomorphism as groups. (In detail, see [3]). \square

2. GROUP E_6

We use the same notation as in [5] (however, some will be rewritten). For example,

- the complex exceptional Jordan algebra $\mathfrak{J}^C = \{X \in M(3, \mathfrak{C}^C) \mid X^* = X\}$, the Freudenthal multiplication $X \times Y$ and the Hermitian inner product $\langle X, Y \rangle$,
- the group $E_6 = \{\alpha \in \text{Iso}_C(\mathfrak{J}^C) \mid \alpha X \times \alpha Y = \tau \alpha \tau(X \times Y), \langle \alpha X, \alpha Y \rangle = \langle X, Y \rangle\}$, and the natural inclusion $F_4 \subset E_6$,

- any element ϕ of the Lie algebra \mathfrak{e}_6 of the group E_6 is uniquely expressed as $\phi = \delta + i\tilde{T}$, $\delta \in \mathfrak{f}_4$, $T \in \mathfrak{J}_0$, where $\mathfrak{J}_0 = \{T \in \mathfrak{J} \mid \text{tr}(T) = 0\}$.

Proposition 2.1. $(E_6)_{E_1} \cong \text{Spin}(10)$.

Proof. We define a 10-dimensional \mathbf{R} -vector space V^{10} by

$$V^{10} = \{X \in \mathfrak{J}^C \mid 2E_1 \times X = -\tau X\} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi & x \\ 0 & \bar{x} & -\tau\xi \end{pmatrix} \mid \xi \in C, x \in \mathfrak{C} \right\}$$

with the norm $1/2\langle X, X \rangle = (\tau\xi)\xi + \bar{x}x$. Let $SO(10) = SO(V^{10})$. Then, we have $(E_6)_{E_1}/\mathbf{Z}_2 \cong SO(10)$, $\mathbf{Z}_2 = \{1, \sigma\}$. Therefore, $(E_6)_{E_1}$ is isomorphic to $\text{Spin}(10)$ as a double covering group of $SO(10)$. (In detail, see [5], [8]). \square

Lemma 2.2. For $\nu \in \text{Spin}(2) = U(1) = \{\nu \in C \mid (\tau\nu)\nu = 1\}$, we define a C -linear transformation $\phi_1(\nu)$ of \mathfrak{J}^C by

$$\phi_1(\nu)X = \begin{pmatrix} \xi_1 & \nu x_3 & \nu^{-1}\bar{x}_2 \\ \nu\bar{x}_3 & \nu^2\xi_2 & x_1 \\ \nu^{-1}x_2 & \bar{x}_1 & \nu^{-2}\xi_3 \end{pmatrix}, \quad X \in \mathfrak{J}^C.$$

Then, $\phi_1(\nu) \in ((E_6)_{E_1})^{\sigma'}$.

Lemma 2.3. Any element ϕ of the Lie algebra $((\mathfrak{e}_6)_{E_1})^{\sigma'}$ of the group $((E_6)_{E_1})^{\sigma'}$ is expressed by

$$\phi = \delta + it(E_2 - E_3)^\sim, \quad \delta \in ((\mathfrak{f}_4)_{E_1})^{\sigma'} = \mathfrak{so}(8), \quad t \in \mathbf{R}.$$

In particular, we have

$$\dim(((\mathfrak{e}_6)_{E_1})^{\sigma'}) = 28 + 1 = 29.$$

Now, we shall determine the group structure of $(\text{Spin}(10))^{\sigma'}$.

Theorem 2.4.

$$(\text{Spin}(10))^{\sigma'} \cong (\text{Spin}(2) \times \text{Spin}(8))/\mathbf{Z}_2, \quad \mathbf{Z}_2 = \{(1, 1), (-1, \sigma)\}.$$

Proof. Let $\text{Spin}(10) = (E_6)_{E_1}$, $\text{Spin}(2) = U(1) \subset ((E_6)_{E_1})^{\sigma'}$ (Lemma 2.2) and $\text{Spin}(8) = ((F_4)_{E_1})^{\sigma'} \subset ((E_6)_{E_1})^{\sigma'}$ (Theorem 1.2, Proposition 2.1). Now, we define a map $\varphi: \text{Spin}(2) \times \text{Spin}(8) \rightarrow (\text{Spin}(10))^{\sigma'}$ by

$$\varphi(\nu, \beta) = \phi_1(\nu)\beta.$$

Then, φ is well-defined: $\varphi(\nu, \beta) \in (\text{Spin}(10))^{\sigma'}$. Since $\phi_1(\nu)$ and β are commutative, φ is a homomorphism. $\text{Ker } \varphi = \{(1, 1), (-1, \sigma)\}$. Since $(\text{Spin}(10))^{\sigma'}$ is connected and $\dim(\mathfrak{spin}(2) \oplus \mathfrak{spin}(8)) = 1 + 28 = 29 = \dim((\mathfrak{spin}(10))^{\sigma'})$ (Lemma 2.3), φ is onto. Thus, we have the isomorphism

$$(\text{Spin}(2) \times \text{Spin}(8))/\mathbf{Z}_2 \cong (\text{Spin}(10))^{\sigma'}. \quad \square$$

3. GROUP E_7

We use the same notation as in [6] (however, some will be rewritten). For example,

- the Freudenthal C -vector space $\mathfrak{P}^C = \mathfrak{J}^C \oplus \mathfrak{J}^C \oplus C \oplus C$, the Hermitian inner product $\langle P, Q \rangle$,
- for $P, Q \in \mathfrak{P}^C$, the C -linear map $P \times Q: \mathfrak{P}^C \rightarrow \mathfrak{P}^C$,
- the group $E_7 = \{\alpha \in \text{Iso}_C(\mathfrak{P}^C) \mid \alpha(P \times Q)\alpha^{-1} = \alpha P \times \alpha Q, \langle \alpha P, \alpha Q \rangle = \langle P, Q \rangle\}$, the natural inclusion $E_6 \subset E_7$ and elements $\sigma, \sigma' \in F_4 \subset E_6 \subset E_7$, $\lambda \in E_7$,
- any element Φ of the Lie algebra \mathfrak{e}_7 of the group E_7 is uniquely expressed as $\Phi = \Phi(\phi, A, -\tau A, \nu)$, $\phi \in \mathfrak{e}_6$, $A \in \mathfrak{J}^C$, $\nu \in i\mathbf{R}$.

In the following, the group $((Spin(10))^{\sigma'})_{F_1(x)}$ is defined by

$$((Spin(10))^{\sigma'})_{F_1(x)} = \{\alpha \in (Spin(10))^{\sigma'} \mid \alpha F_1(x) = F_1(x) \text{ for all } x \in \mathfrak{C}\},$$

$$\text{where } F_1(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & \bar{x} & 0 \end{pmatrix} \in \mathfrak{J}.$$

Proposition 3.1. $((Spin(10))^{\sigma'})_{F_1(x)} \cong Spin(2)$.

Proof. Let $Spin(10) = (E_6)_{E_1}$ and $Spin(2) = U(1) = \{\nu \in C \mid (\tau\nu)\nu = 1\}$. We consider the map $\phi_1: Spin(2) \rightarrow ((Spin(10))^{\sigma'})_{F_1(x)}$ defined in Section 2. Then, ϕ_1 is well-defined: $\phi_1(\nu) \in ((Spin(10))^{\sigma'})_{F_1(x)}$. We shall show that ϕ_1 is onto. From $((Spin(10))^{\sigma'})_{F_1(x)} \subset (Spin(10))^{\sigma'}$, we see that for $\alpha \in ((Spin(10))^{\sigma'})_{F_1(x)}$, there exist $\nu \in Spin(2)$ and $\beta \in Spin(8)$ such that $\alpha = \varphi(\nu, \beta)$ (Theorem 2.4). Further, from $\alpha F_1(x) = F_1(x)$ and $\phi_1(\nu)F_1(x) = F_1(x)$, we have $\beta F_1(x) = F_1(x)$. Hence, $\beta = (1, 1, 1)$ or $(1, -1, -1) = \sigma$ by the principle of triality. Hence, $\alpha = \phi_1(\nu)$ or $\phi_1(\nu)\sigma$. However, in the latter case, from $\sigma = \phi_1(-1)$, we have $\alpha = \phi_1(\nu)\phi_1(-1) = \phi_1(-\nu)$. Therefore, ϕ_1 is onto. $\text{Ker } \phi_1 = \{1\}$. Thus, we have the isomorphism

$$Spin(2) \cong ((Spin(10))^{\sigma'})_{F_1(x)}. \quad \square$$

We define C -linear maps $\kappa, \mu: \mathfrak{P}^C \rightarrow \mathfrak{P}^C$ respectively by

$$\begin{aligned} \kappa(X, Y, \xi, \eta) &= (-\kappa_1 X, \kappa_1 Y, -\xi, \eta), \quad \kappa_1 X = (E_1, X)E_1 - 4E_1 \times (E_1 \times X), \\ \mu(X, Y, \xi, \eta) &= (2E_1 \times Y + \eta E_1, 2E_1 \times X + \xi E_1, (E_1, Y), (E_1, X)). \end{aligned}$$

Their explicit forms are

$$\begin{aligned}\kappa(X, Y, \xi, \eta) &= \left(\begin{pmatrix} -\xi_1 & 0 & 0 \\ 0 & \xi_2 & x_1 \\ 0 & \bar{x}_1 & \xi_3 \end{pmatrix}, \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & -\eta_2 & -y_1 \\ 0 & -\bar{y}_1 & -\eta_3 \end{pmatrix}, -\xi, \eta \right), \\ \mu(X, Y, \xi, \eta) &= \left(\begin{pmatrix} \eta & 0 & 0 \\ 0 & \eta_3 & -y_1 \\ 0 & -\bar{y}_1 & \eta_2 \end{pmatrix}, \begin{pmatrix} \xi & 0 & 0 \\ 0 & \xi_3 & -x_1 \\ 0 & -\bar{x}_1 & \xi_2 \end{pmatrix}, \eta_1, \xi_1 \right).\end{aligned}$$

We define subgroup $(E_7)^{\kappa, \mu}$ of E_7 by

$$(E_7)^{\kappa, \mu} = \{\alpha \in E_7 \mid \kappa\alpha = \alpha\kappa, \mu\alpha = \alpha\mu\},$$

and also define subgroups $((E_7)^{\kappa, \mu})_{(0, E_1, 0, 1)}$, $((E_7)^{\kappa, \mu})_{(0, E_1, 0, 1), (0, -E_1, 0, 1)}$, $((E_7)^{\kappa, \mu})_{(E_1, 0, 1, 0)}$ and $((E_7)^{\kappa, \mu})_{(E_1, 0, 1, 0), (E_1, 0, -1, 0)}$ of E_7 by

$$\begin{aligned}((E_7)^{\kappa, \mu})_{(0, E_1, 0, 1)} &= \{\alpha \in (E_7)^{\kappa, \mu} \mid \alpha(0, E_1, 0, 1) = (0, E_1, 0, 1)\}, \\ ((E_7)^{\kappa, \mu})_{(0, E_1, 0, 1), (0, -E_1, 0, 1)} &= \left\{ \alpha \in (E_7)^{\kappa, \mu} \left| \begin{array}{l} \alpha(0, E_1, 0, 1) = (0, E_1, 0, 1) \\ \alpha(0, -E_1, 0, 1) = (0, -E_1, 0, 1) \end{array} \right. \right\}, \\ ((E_7)^{\kappa, \mu})_{(E_1, 0, 1, 0)} &= \{\alpha \in (E_7)^{\kappa, \mu} \mid \alpha(E_1, 0, 1, 0) = (E_1, 0, 1, 0)\}, \\ ((E_7)^{\kappa, \mu})_{(E_1, 0, 1, 0), (E_1, 0, -1, 0)} &= \left\{ \alpha \in (E_7)^{\kappa, \mu} \left| \begin{array}{l} \alpha(E_1, 0, 1, 0) = (E_1, 0, 1, 0) \\ \alpha(E_1, 0, -1, 0) = (E_1, 0, -1, 0) \end{array} \right. \right\}.\end{aligned}$$

Proposition 3.2. (1) $((E_7)^{\kappa, \mu})_{(E_1, 0, 1, 0)} = ((E_7)^{\kappa, \mu})_{(0, E_1, 0, 1)}$.

(2) $((E_7)^{\kappa, \mu})_{(E_1, 0, 1, 0), (E_1, 0, -1, 0)} = ((E_7)^{\kappa, \mu})_{(0, E_1, 0, 1), (0, -E_1, 0, 1)}$.

Proof. (1) For $\alpha \in ((E_7)^{\kappa, \mu})_{(E_1, 0, 1, 0)}$, we have

$$\alpha(0, E_1, 0, 1) = \alpha\mu(E_1, 0, 1, 0) = \mu\alpha(E_1, 0, 1, 0) = \mu(E_1, 0, 1, 0) = (0, E_1, 0, 1).$$

Hence, $\alpha \in ((E_7)^{\kappa, \mu})_{(0, E_1, 0, 1)}$. The converse is also proved.

(2) It is proved in a way similar to (1). □

Proposition 3.3. $((E_7)^{\kappa, \mu})_{(0, E_1, 0, 1), (0, -E_1, 0, 1)} \cong Spin(10)$.

Proof. If $\alpha \in E_7$ satisfies $\alpha(0, E_1, 0, 1) = (0, E_1, 0, 1)$ and $\alpha(0, -E_1, 0, 1) = (0, -E_1, 0, 1)$, then we have $\alpha(0, 0, 0, 1) = (0, 0, 0, 1)$ and $\alpha(0, E_1, 0, 0) = (0, E_1, 0, 0)$. From the first condition, we see that $\alpha \in E_6$. Moreover, from the second condition, we have $\alpha \in (E_6)_{E_1} = Spin(10)$. The proof of the converse is trivial because κ, μ are defined by using E_1 . □

Proposition 3.4. $((E_7)^{\kappa,\mu})_{(0,E_1,0,1)} \cong Spin(11)$.

Proof. We define an 11-dimensional \mathbf{R} -vector space V^{11} by

$$V^{11} = \{P \in \mathfrak{P}^C \mid \kappa P = P, \mu\tau\lambda P = P, P \times (0, E_1, 0, 1) = 0\} \\ = \left\{ \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi & x \\ 0 & \bar{x} & -\tau\xi \end{pmatrix}, \begin{pmatrix} \eta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \tau\eta \right) \mid x \in \mathfrak{C}, \xi \in C, \eta \in i\mathbf{R} \right\}$$

with the norm

$$(P, P)_\mu = \frac{1}{2}(\mu P, \lambda P) = (\tau\eta)\eta + \bar{x}x + (\tau\xi)\xi.$$

Let $SO(11) = SO(V^{11})$. Then, we have $((E_7)^{\kappa,\mu})_{(0,E_1,0,1)}/\mathbf{Z}_2 \cong SO(11)$, $\mathbf{Z}_2 = \{1, \sigma\}$. Therefore, $((E_7)^{\kappa,\mu})_{(0,E_1,0,1)}$ is isomorphic to $Spin(11)$ as a double covering group of $SO(11)$. (In detail, see [6], [8]). \square

Now, we shall consider the following group

$$((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)} \\ = \{\alpha \in (Spin(11))^{\sigma'} \mid \alpha(0, F_1(y), 0, 0) = (0, F_1(y), 0, 0) \text{ for all } y \in \mathfrak{C}\}.$$

Lemma 3.5. The Lie algebra $((\mathfrak{spin}(11))^{\sigma'})_{(0,F_1(y),0,0)}$ of the group $((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)}$ is given by

$$((\mathfrak{spin}(11))^{\sigma'})_{(0,F_1(y),0,0)} \\ = \left\{ \Phi \left(i \begin{pmatrix} 0 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & -\epsilon \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \tau\rho \end{pmatrix}, -\tau \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \tau\rho \end{pmatrix}, 0 \right) \mid \epsilon \in \mathbf{R}, \rho \in C \right\}.$$

In particular, we have

$$\dim(((\mathfrak{spin}(11))^{\sigma'})_{(0,F_1(y),0,0)}) = 3.$$

Lemma 3.6. For $a \in \mathbf{R}$, the maps $\alpha_k(a): \mathfrak{P}^C \rightarrow \mathfrak{P}^C$, $k = 1, 2, 3$ defined by

$$\alpha_k(a) \begin{pmatrix} X \\ Y \\ \xi \\ \eta \end{pmatrix} = \begin{pmatrix} (1 + (\cos a - 1)p_k)X - 2(\sin a)E_k \times Y + \eta(\sin a)E_k \\ 2(\sin a)E_k \times X + (1 + (\cos a - 1)p_k)Y - \xi(\sin a)E_k \\ ((\sin a)E_k, Y) + (\cos a)\xi \\ -(\sin a)E_k, X + (\cos a)\eta \end{pmatrix}$$

belong to the group E_7 , where $p_k: \mathfrak{J}^C \rightarrow \mathfrak{J}^C$ is defined by

$$p_k(X) = (X, E_k)E_k + 4E_k \times (E_k \times X), \quad X \in \mathfrak{J}^C.$$

$\alpha_1(a), \alpha_2(b), \alpha_3(c)$ ($a, b, c \in \mathbf{R}$) commute with each other.

Proof. For $\Phi_k(a) = \Phi(0, aE_k, -aE_k, 0) \in \mathfrak{e}_7$, we have $\alpha_k(a) = \exp \Phi_k(a) \in E_7$. Since $[\Phi_k(a), \Phi_l(b)] = 0$, $k \neq l$, $\alpha_k(a)$ and $\alpha_l(b)$ are commutative. \square

Lemma 3.7. $((Spin(11))^{\sigma'})_{(0, F_1(y), 0, 0)} / Spin(2) \simeq S^2$.

In particular, $((Spin(11))^{\sigma'})_{(0, F_1(y), 0, 0)}$ is connected.

Proof. We define a 3-dimensional \mathbf{R} -vector space W^3 by

$$W^3 = \{P \in \mathfrak{P}^C \mid \kappa P = -P, \mu\tau\lambda P = -P, \sigma'P = P, P \times (E_1, 0, 1, 0) = 0\} \\ = \left\{ P = \left(\begin{pmatrix} i\xi & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & -\tau\eta \end{pmatrix}, -i\xi, 0 \right) \mid \xi \in \mathbf{R}, \eta \in C \right\}$$

with the norm

$$(P, P)_\mu = -\frac{1}{2}(\mu P, \lambda P) = \xi^2 + (\tau\eta)\eta.$$

Then, $S^2 = \{P \in W^3 \mid (P, P)_\mu = 1\}$ is a 2-dimensional sphere. The group $((Spin(11))^{\sigma'})_{(0, F_1(y), 0, 0)}$ acts on S^2 . We shall show that this action is transitive. To show this, it is sufficient to show that any element $P \in S^2$ can be transformed to $(-iE_1, 0, i, 0) \in S^2$ under the action of $((Spin(11))^{\sigma'})_{(0, F_1(y), 0, 0)}$. Now, for a given

$$P = \left(\begin{pmatrix} i\xi & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & -\tau\eta \end{pmatrix}, -i\xi, 0 \right) \in S^2,$$

choose $a \in \mathbf{R}$, $0 \leq a < \pi/2$ such that $\tan 2a = -\frac{2i\xi}{\tau\eta - \eta}$ (if $\tau\eta - \eta = 0$, then let $a = \pi/4$). Operate $\alpha_{23}(a) := \alpha_2(a)\alpha_3(a) = \exp(\Phi(0, a(E_2 + E_3), -a(E_2 + E_3), 0)) \in ((Spin(11))^{\sigma'})_{(0, F_1(y), 0, 0)}$ (Lemmas 3.5, 3.6) on P . Then, we have the ξ -term of $\alpha_{23}(a)P$ is $-(\cos 2a)(i\xi) + 1/2(\sin 2a)(\tau\eta - \eta) = 0$. Hence,

$$\alpha_{23}(a)P = \left(0, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & -\tau\zeta \end{pmatrix}, 0, 0 \right) = P_1, \quad \zeta \in C, \quad (\tau\zeta)\zeta = 1.$$

From $(\tau\zeta)\zeta = 1$, $\zeta \in C$, we can put $\zeta = e^{i\theta}$, $0 \leq \theta < 2\pi$. Let $\nu = e^{-i\theta/2}$, and operate $\phi_1(\nu) \in ((Spin(10))^{\sigma'})_{F_1(x)}$ (Lemma 2.2) ($\subset ((Spin(11))^{\sigma'})_{(0, F_1(x), 0, 0)}$) on P_1 . Then,

$$\phi_1(\nu)P_1 = (0, E_2 - E_3, 0, 0) = P_2.$$

Moreover, operate $\phi_1(e^{i\pi/4})$ on P_2 ,

$$\phi_1(e^{i\pi/4})P_2 = (0, i(E_2 + E_3), 0, 0) = P_3.$$

Operate again $\alpha_{23}(\pi/4)$ on P_3 . Then, we have

$$\alpha_{23}(\pi/4)P_3 = (-iE_1, 0, i, 0).$$

This shows the transitivity. The isotropy subgroup of $((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)}$ at $(-iE_1, 0, i, 0)$ is $((Spin(10))^{\sigma'})_{F_1(y)}$ (Propositions 3.2 (2), 3.3, 3.4) = $Spin(2)$. Thus, we have the homeomorphism

$$((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)}/Spin(2) \simeq S^2. \quad \square$$

Proposition 3.8. $((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)} \cong Spin(3)$.

Proof. Since $((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)}$ is connected (Lemma 3.7), we can define a homomorphism $\pi: ((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)} \rightarrow SO(3) = SO(W^3)$ by

$$\pi(\alpha) = \alpha|W^3.$$

$\text{Ker } \pi = \{1, \sigma\} = \mathbf{Z}_2$. Since $\dim(((\mathfrak{spin}(11))^{\sigma'})_{(0,F_1(y),0,0)}) = 3$ (Lemma 3.5) = $\dim(\mathfrak{so}(3))$, π is onto. Hence, $((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)}/\mathbf{Z}_2 \cong SO(3)$. Therefore, $((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)}$ is isomorphic to $Spin(3)$ as a double covering group of $SO(3)$. \square

Lemma 3.9. The Lie algebra $(\mathfrak{spin}(11))^{\sigma'}$ of the group $(Spin(11))^{\sigma'}$ is given by

$$\begin{aligned} & (\mathfrak{spin}(11))^{\sigma'} \\ &= \left\{ \Phi \left(D + i \begin{pmatrix} 0 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & -\epsilon \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \tau\rho \end{pmatrix}, -\tau \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \tau\rho \end{pmatrix}, 0 \right) \right. \\ & \quad \left. \left| D \in \mathfrak{so}(8), \epsilon \in \mathbf{R}, \rho \in C \right\}. \end{aligned}$$

In particular, we have

$$\dim((\mathfrak{spin}(11))^{\sigma'}) = 28 + 3 = 31.$$

Now, we shall determine the group structure of $(Spin(11))^{\sigma'}$.

Theorem 3.10.

$$(Spin(11))^{\sigma'} \cong (Spin(3) \times Spin(8))/\mathbf{Z}_2, \quad \mathbf{Z}_2 = \{(1, 1), (-1, \sigma)\}.$$

Proof. Let $Spin(11) = ((E_7)^{\kappa,\mu})_{(0,E_1,0,1)}$, $Spin(3) = ((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)}$ and $Spin(8) = ((F_4)_{E_1})^{\sigma'} \subset ((E_6)_{E_1})^{\sigma'} = (((E_7)^{\kappa,\mu})_{(E_1,0,1,0),(E_1,0,-1,0)})^{\sigma'} \subset (((E_7)^{\kappa,\mu})_{(E_1,0,1,0)})^{\sigma'}$ (Theorem 1.2, Propositions 3.2, 3.3, 3.4). Now, we define a map $\varphi: Spin(3) \times Spin(8) \rightarrow (Spin(11))^{\sigma'}$ by

$$\varphi(\alpha, \beta) = \alpha\beta.$$

Then, φ is well-defined: $\varphi(\alpha, \beta) \in (Spin(11))^{\sigma'}$. Since $[\Phi_D, \Phi_3] = 0$ for $\Phi_D = \Phi(D, 0, 0, 0) \in \mathfrak{spin}(8)$, $\Phi_3 \in \mathfrak{spin}(3) = ((\mathfrak{spin}(11))^{\sigma'})_{(0, F_1(y), 0, 0)}$ (Proposition 3.8), we have $\alpha\beta = \beta\alpha$. Hence, φ is a homomorphism. $\text{Ker } \varphi = \{(1, 1), (-1, \sigma)\} = \mathbf{Z}_2$. Since $(Spin(11))^{\sigma'}$ is connected and $\dim(\mathfrak{spin}(3) \oplus \mathfrak{spin}(8)) = 3$ (Lemma 3.5) $+28 = 31 = \dim((\mathfrak{spin}(11))^{\sigma'})$ (Lemma 3.9), φ is onto. Thus, we have the isomorphism

$$(Spin(3) \times Spin(8))/\mathbf{Z}_2 \cong (Spin(11))^{\sigma'}. \quad \square$$

Proposition 3.11. $(E_7)^{\kappa, \mu} \cong Spin(12)$.

Proof. We define a 12-dimensional \mathbf{R} -vector space V^{12} by

$$\begin{aligned} V^{12} &= \{P \in \mathfrak{P}^C \mid \kappa P = P, \mu\tau\lambda P = P\} \\ &= \left\{ \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \xi & x \\ 0 & \bar{x} & -\tau\xi \end{pmatrix}, \begin{pmatrix} \eta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \tau\eta \right) \mid x \in \mathfrak{C}, \xi, \eta \in C \right\} \end{aligned}$$

with the norm

$$(P, P)_\mu = \frac{1}{2}(\mu P, \lambda P) = (\tau\eta)\eta + \bar{x}x + (\tau\xi)\xi.$$

Let $SO(12) = SO(V^{12})$. Then, we have $(E_7)^{\kappa, \mu}/\mathbf{Z}_2 \cong SO(12)$, $\mathbf{Z}_2 = \{1, \sigma\}$. Therefore, $(E_7)^{\kappa, \mu}$ is isomorphic to $Spin(12)$ as a double covering group of $SO(12)$. (In detail, see [6], [8]). \square

Now, we shall consider the following group

$$\begin{aligned} &((Spin(12))^{\sigma'})_{(0, F_1(y), 0, 0)} \\ &= \{\alpha \in (Spin(12))^{\sigma'} \mid \alpha(0, F_1(y), 0, 0) = (0, F_1(y), 0, 0) \text{ for all } y \in \mathfrak{C}\}. \end{aligned}$$

Lemma 3.12. *The Lie algebra $((\mathfrak{spin}(12))^{\sigma'})_{(0, F_1(y), 0, 0)}$ of the group $((Spin(12))^{\sigma'})_{(0, F_1(y), 0, 0)}$ is given by*

$$\begin{aligned} &((\mathfrak{spin}(12))^{\sigma'})_{(0, F_1(y), 0, 0)} \\ &= \left\{ \Phi \left(i \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix} \sim, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{pmatrix}, -\tau \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{pmatrix}, -\frac{3}{2}i\epsilon_1 \right) \right. \\ &\quad \left. \mid \epsilon_i \in \mathbf{R}, \epsilon_1 + \epsilon_2 + \epsilon_3 = 0, \rho_i \in C \right\}. \end{aligned}$$

In particular, we have

$$\dim(((\mathfrak{spin}(12))^{\sigma'})_{(0, F_1(y), 0, 0)}) = 6.$$

Lemma 3.13. For $t \in \mathbf{R}$, the map $\alpha(t): \mathfrak{P}^C \rightarrow \mathfrak{P}^C$ defined by

$$\alpha(t)(X, Y, \xi, \eta) = \left(\begin{pmatrix} e^{2it}\xi_1 & e^{it}x_3 & e^{it}\bar{x}_2 \\ e^{it}\bar{x}_3 & \xi_2 & x_1 \\ e^{it}x_2 & \bar{x}_1 & \xi_3 \end{pmatrix}, \begin{pmatrix} e^{-2it}\eta_1 & e^{-it}y_3 & e^{-it}\bar{y}_2 \\ e^{-it}\bar{y}_3 & \eta_2 & y_1 \\ e^{-it}y_2 & \bar{y}_1 & \eta_3 \end{pmatrix}, e^{-2it}\xi, e^{2it}\eta \right)$$

belongs to the group $((Spin(12))^{\sigma'})_{(0, F_1(y), 0, 0)}$.

Proof. For $\Phi = \Phi(2itE_1 \vee E_1, 0, 0, -2it) \in ((\mathfrak{spin}(12))^{\sigma'})_{(0, F_1(y), 0, 0)}$ (Lemma 3.12), we have $\alpha(t) = \exp \Phi \in ((Spin(12))^{\sigma'})_{(0, F_1(y), 0, 0)}$. \square

Lemma 3.14. $((Spin(12))^{\sigma'})_{(0, F_1(y), 0, 0)} / Spin(3) \simeq S^3$.

In particular, $((Spin(12))^{\sigma'})_{(0, F_1(y), 0, 0)}$ is connected.

Proof. We define a 4-dimensional \mathbf{R} -vector space W^4 by

$$W^4 = \{P \in \mathfrak{P}^C \mid \kappa P = -P, \mu\tau\lambda P = -P, \sigma'P = P\} \\ = \left\{ P = \left(\begin{pmatrix} \xi & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & -\tau\eta \end{pmatrix}, \tau\xi, 0 \right) \mid \xi, \eta \in C \right\}$$

with the norm

$$(P, P)_\mu = -\frac{1}{2}(\mu P, \lambda P) = (\tau\xi)\xi + (\tau\eta)\eta.$$

Then, $S^3 = \{P \in W^4 \mid (P, P)_\mu = 1\}$ is a 3-dimensional sphere. The group $((Spin(12))^{\sigma'})_{(0, F_1(y), 0, 0)}$ acts on S^3 . We shall show that this action is transitive. To show this, it is sufficient to show that any element $P \in S^3$ can be transformed to $(E_1, 0, 1, 0) \in S^3$ under the action of $((Spin(12))^{\sigma'})_{(0, F_1(y), 0, 0)}$. Now, for a given

$$P = \left(\begin{pmatrix} \xi & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & -\tau\eta \end{pmatrix}, \tau\xi, 0 \right) \in S^3,$$

choose $t \in \mathbf{R}$ such that $e^{2it}\xi \in i\mathbf{R}$. Operate $\alpha(t)$ (Lemma 3.13) on P . Then, we have

$$\alpha(t)P = P_1 \in S^2 \subset S^3.$$

Now, since $((Spin(11))^{\sigma'})_{(0, F_1(y), 0, 0)} \subset ((Spin(12))^{\sigma'})_{(0, F_1(y), 0, 0)}$ acts transitively on S^2 (Lemma 3.7), there exists $\beta \in ((Spin(11))^{\sigma'})_{(0, F_1(y), 0, 0)}$ such that

$$\beta P_1 = (-iE_1, 0, i, 0) = P_2.$$

Operate again $\alpha(\pi/4)$ on P_2 . Then, we have

$$\alpha(\pi/4)P_2 = (E_1, 0, 1, 0).$$

This shows the transitivity. The isotropy subgroup of $((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)}$ at $(E_1, 0, 1, 0)$ is $((Spin(11))^{\sigma'})_{(0,F_1(y),0,0)}$ (Propositions 3.2 (1), 3.4, 3.11) $= Spin(3)$. Thus, we have the homeomorphism

$$((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)}/Spin(3) \simeq S^3. \quad \square$$

Proposition 3.15. $((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)} \cong Spin(4)$.

Proof. Since $((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)}$ is connected (Lemma 3.14), we can define a homomorphism $\pi: ((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)} \rightarrow SO(4) = SO(W^4)$ by

$$\pi(\alpha) = \alpha|W^4.$$

$\text{Ker } \pi = \{1, \sigma\} = \mathbf{Z}_2$. Since $\dim((\mathfrak{spin}(12))^{\sigma'})_{(0,F_1(y),0,0)} = 6$ (Lemma 3.12) $= \dim(\mathfrak{so}(4))$, π is onto. Hence, $((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)}/\mathbf{Z}_2 \cong SO(4)$. Therefore, $((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)}$ is isomorphic to $Spin(4)$ as a double covering group of $SO(4)$. \square

Lemma 3.16. *The Lie algebra $(\mathfrak{spin}(12))^{\sigma'}$ of the group $(Spin(12))^{\sigma'}$ is given by*

$$\begin{aligned} & (\mathfrak{spin}(12))^{\sigma'} \\ &= \left\{ \Phi \left(D + i \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{pmatrix}, -\tau \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{pmatrix}, -i \frac{3}{2} \epsilon_1 \right) \right. \\ & \quad \left. \left| D \in \mathfrak{so}(8), \epsilon_i \in \mathbf{R}, \epsilon_1 + \epsilon_2 + \epsilon_3 = 0, \rho_i \in C \right\}. \end{aligned}$$

In particular, we have

$$\dim((\mathfrak{spin}(12))^{\sigma'}) = 28 + 6 = 34.$$

Now, we shall determine the group structure of $(Spin(12))^{\sigma'}$.

Theorem 3.17.

$$(Spin(12))^{\sigma'} \cong (Spin(4) \times Spin(8))/\mathbf{Z}_2, \quad \mathbf{Z}_2 = \{(1, 1), (-1, \sigma)\}.$$

Proof. Let $Spin(12) = (E_7)^{\kappa, \mu}$, $Spin(4) = ((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)}$ and $Spin(8) = ((F_4)_{E_1})^{\sigma'} \subset ((E_6)_{E_1})^{\sigma'} = (((E_7)^{\kappa, \mu})_{(E_1,0,1,0),(E_1,0,-1,0)})^{\sigma'} \subset ((E_7)^{\kappa, \mu})^{\sigma'}$ (Theorem 1.2, Propositions 3.2, 3.3, 3.11, 3.15). Now, we define a map $\varphi: Spin(4) \times Spin(8) \rightarrow (Spin(12))^{\sigma'}$ by

$$\varphi(\alpha, \beta) = \alpha\beta.$$

Then, φ is well-defined: $\varphi(\alpha, \beta) \in (Spin(12))^{\sigma'}$. Since $[\Phi_D, \Phi_4] = 0$ for $\Phi_D = \Phi(D, 0, 0, 0) \in \mathfrak{spin}(8)$, $\Phi_4 \in \mathfrak{spin}(4) = ((\mathfrak{spin}(12))^{\sigma'})_{(0, F_1(y), 0, 0)}$ (Proposition 3.15), we have $\alpha\beta = \beta\alpha$. Hence, φ is a homomorphism. $\text{Ker } \varphi = \{(1, 1), (-1, \sigma)\} = \mathbf{Z}_2$. Since $(Spin(12))^{\sigma'}$ is connected and $\dim(\mathfrak{spin}(4) \oplus \mathfrak{spin}(8)) = 6$ (Lemma 3.12) $+ 28 = 34 = \dim((\mathfrak{spin}(12))^{\sigma'})$ (Lemma 3.16), φ is onto. Thus, we have the isomorphism

$$(Spin(4) \times Spin(8))/\mathbf{Z}_2 \cong (Spin(12))^{\sigma'}. \quad \square$$

4. GROUP E_8

We use the same notation as in [2], [4] (however, some will be rewritten). For example,

- C -Lie algebra $\mathfrak{e}_8^C = \mathfrak{e}_7^C \oplus \mathfrak{P}^C \oplus \mathfrak{P}^C \oplus C \oplus C \oplus C$ and C -linear transformations $\lambda, \tilde{\lambda}$ of \mathfrak{e}_8^C ,
- the groups $E_8^C = \{\alpha \in \text{Iso}_C(\mathfrak{e}_8^C) \mid \alpha[R_1, R_2] = [\alpha R_1, \alpha R_2]\}$ and $E_8 = (E_8^C)^{\tau\tilde{\lambda}} = \{\alpha \in E_8^C \mid \tau\tilde{\lambda}\alpha = \alpha\tau\tilde{\lambda}\}$.

For $\alpha \in E_7$, the map $\tilde{\alpha}: \mathfrak{e}_8^C \rightarrow \mathfrak{e}_8^C$ is defined by

$$\tilde{\alpha}(\Phi, P, Q, r, u, v) = (\alpha\Phi\alpha^{-1}, \alpha P, \alpha Q, r, u, v).$$

Then, $\tilde{\alpha} \in E_8$ and we identify α with $\tilde{\alpha}$. The group E_8 contains E_7 as a subgroup by

$$E_7 = \{\tilde{\alpha} \in E_8 \mid \alpha \in E_7\} = (E_8)_{(0,0,0,0,1,0)}.$$

We define a C -linear map $\tilde{\kappa}: \mathfrak{e}_8^C \rightarrow \mathfrak{e}_8^C$ by

$$\tilde{\kappa} = \text{ad}(\kappa, 0, 0, -1, 0, 0) = \text{ad}(\Phi(-2E_1 \vee E_1, 0, 0, -1), 0, 0, -1, 0, 0),$$

and 14-dimensional C -vector spaces \mathfrak{g}_{-2} and \mathfrak{g}_2 by

$$\begin{aligned} \mathfrak{g}_{-2} &= \{R \in \mathfrak{e}_8^C \mid \tilde{\kappa}R = -2R\} \\ &= \{(\Phi(0, \zeta E_1, 0, 0), (\xi_1 E_1, \eta_2 E_2 + \eta_3 E_3 + F_1(y), \xi, 0), 0, 0, u, 0) \\ &\quad \mid \zeta, \xi_1, \eta_i, \xi, u \in C, y \in \mathfrak{C}^C\}, \\ \mathfrak{g}_2 &= \{R \in \mathfrak{e}_8^C \mid \tilde{\kappa}R = 2R\} \\ &= \{(\Phi(0, 0, \zeta E_1, 0), 0, (\xi_2 E_1 + \xi_3 E_3 + F_1(x), \eta_1 E_1, 0, \eta), 0, 0, v) \\ &\quad \mid \zeta, \xi_i, \eta_1, \eta, v \in C, x \in \mathfrak{C}^C\}. \end{aligned}$$

Further, we define two C -linear maps $\tilde{\mu}_1: \mathfrak{e}_8^C \rightarrow \mathfrak{e}_8^C$ and $\delta: \mathfrak{g}_2 \rightarrow \mathfrak{g}_2$ by

$$\tilde{\mu}_1(\Phi, P, Q, r, u, v) = (\mu_1\Phi\mu_1^{-1}, i\mu_1Q, i\mu_1P, -r, v, u),$$

where

$$\mu_1(X, Y, \xi, \eta) = \left(\begin{pmatrix} i\eta & x_3 & \bar{x}_2 \\ \bar{x}_3 & i\eta_3 & -iy_1 \\ x_2 & -i\bar{y}_1 & i\eta_2 \end{pmatrix}, \begin{pmatrix} i\xi & y_3 & \bar{y}_2 \\ \bar{y}_3 & i\xi_3 & -ix_1 \\ y_2 & -i\bar{x}_1 & i\xi_2 \end{pmatrix}, i\eta_1, i\xi_1 \right),$$

and

$$\begin{aligned} & \delta(\Phi(0, 0, \zeta E_1, 0), 0, (\xi_2 E_2 + \xi_3 E_3 + F_1(x), \eta_1 E_1, 0, \eta), 0, 0, v) \\ &= (\Phi(0, 0, -v E_1, 0), 0, (\xi_2 E_2 + \xi_3 E_3 + F_1(x), \eta_1 E_1, 0, \eta), 0, 0, -\zeta). \end{aligned}$$

In particular, the explicit form of the map $\tilde{\mu}_1: \mathfrak{g}_{-2} \rightarrow \mathfrak{g}_2$ is given by

$$\begin{aligned} & \tilde{\mu}_1(\Phi(0, \zeta E_1, 0, 0), (\xi_1 E_1, \eta_2 E_2 + \eta_3 E_3 + F_1(y), \xi, 0), 0, 0, u, 0) \\ &= (\Phi(0, 0, \zeta E_1, 0), 0, (-\eta_3 E_2 - \eta_2 E_3 + F_1(y), -\xi E_1, 0, -\xi_1), 0, 0, u). \end{aligned}$$

The composition map $\delta\tilde{\mu}_1: \mathfrak{g}_{-2} \rightarrow \mathfrak{g}_2$ of $\tilde{\mu}_1$ and $\delta\tilde{\mu}_1$ is denoted by $\tilde{\mu}_\delta$:

$$\begin{aligned} & \tilde{\mu}_\delta(\Phi(0, \zeta E_1, 0, 0), (\xi_1 E_1, \eta_2 E_2 + \eta_3 E_3 + F_1(y), \xi, 0), 0, 0, u, 0) \\ &= (\Phi(0, 0, -u E_1, 0), 0, (-\eta_3 E_2 - \eta_2 E_3 + F_1(y), -\xi E_1, 0, -\xi_1), 0, 0, -\zeta). \end{aligned}$$

Now, we define the inner product $(R_1, R_2)_\mu$ in \mathfrak{g}_{-2} by

$$(R_1, R_2)_\mu = \frac{1}{30} B_8(\tilde{\mu}_\delta R_1, R_2),$$

where B_8 is the Killing form of \mathfrak{e}_8^C . The explicit form of $(R, R)_\mu$ is given by

$$(R, R)_\mu = -4\zeta u - \eta_2 \eta_3 + \bar{y}y + \xi_1 \xi$$

for $R = (\Phi(0, \zeta E_1, 0, 0), (\xi_1 E_1, \eta_2 E_2 + \eta_3 E_3 + F_1(y), \xi, 0), 0, 0, u, 0) \in \mathfrak{g}_{-2}$.

Hereafter, we use the notation $(V^C)^{14}$ instead of \mathfrak{g}_{-2} .

We define **R**-vector spaces V^{14} , V^{13} and $(V')^{12}$ respectively by

$$\begin{aligned} V^{14} &= \{R \in (V^C)^{14} \mid \tilde{\mu}_\delta \tau \bar{\lambda} R = -R\} \\ &= \{R = (\Phi(0, \zeta E_1, 0, 0), (\xi E_1, \eta E_2 - \tau \eta E_3 + F_1(y), \tau \xi, 0), 0, 0, -\tau \zeta, 0) \\ &\quad \mid \zeta, \xi, \eta \in C, y \in \mathfrak{C}\} \end{aligned}$$

with the norm

$$(R, R)_\mu = \frac{1}{30} B_8(\tilde{\mu}_\delta R, R) = 4(\tau \zeta) \zeta + (\tau \eta) \eta + \bar{y}y + (\tau \xi) \xi,$$

$$\begin{aligned} V^{13} &= \{R \in V^{14} \mid (R, (\Phi_1, 0, 0, 0, 1, 0))_\mu = 0\} \\ &= \{R = (\Phi(0, \zeta E_1, 0, 0), (\xi E_1, \eta E_2 - \tau \eta E_3 + F_1(y), \tau \xi, 0), 0, 0, -\zeta, 0) \\ &\quad \mid \zeta \in \mathbf{R}, \xi, \eta \in C, y \in \mathfrak{C}\} \end{aligned}$$

with the norm

$$(R, R)_\mu = \frac{1}{30} B_8(\tilde{\mu}_\delta R, R) = 4\zeta^2 + (\tau \eta) \eta + \bar{y}y + (\tau \xi) \xi,$$

$$\begin{aligned}(V')^{12} &= \{R \in V^{13} \mid (R, (\Phi_1, 0, 0, 0, -1, 0))_\mu = 0\} \\ &= \{R = (0, (\xi E_1, \eta E_2 - \tau \eta E_3 + F_1(y), \tau \xi, 0), 0, 0, 0, 0) \\ &\quad \mid \xi, \eta \in C, y \in \mathfrak{C}\}\end{aligned}$$

with the norm

$$(R, R)_\mu = \frac{1}{30} B_8(\tilde{\mu}_\delta R, R) = (\tau \eta) \eta + \bar{y} y + (\tau \xi) \xi,$$

where $\Phi_1 = \Phi(0, E_1, 0, 0)$. We use the notation $(V')^{12}$ to distinguish from the \mathbf{R} -vector space V^{12} defined in Section 3. The space $(V')^{12}$ above can be identified with the \mathbf{R} -vector space

$$\begin{aligned}&\{P \in \mathfrak{P}^C \mid \kappa P = -P, \mu \tau \lambda P = -P\} \\ &= \{P = (\xi E_1, \eta E_2 - \tau \eta E_3 + F_1(y), \tau \xi, 0) \in \mathfrak{P}^C \mid \xi, \eta \in C, y \in \mathfrak{C}\}\end{aligned}$$

with the norm

$$(P, P)_\mu = -\frac{1}{2}(\mu P, \lambda P) = (\tau \eta) \eta + \bar{y} y + (\tau \xi) \xi.$$

Now, we define a subgroup G_{14} of E_8^C by

$$G_{14} = \{\alpha \in E_8^C \mid \tilde{\kappa} \alpha = \alpha \tilde{\kappa}, \tilde{\mu}_\delta \alpha R = \alpha \tilde{\mu}_\delta R, R \in (V^C)^{14}\}.$$

Lemma 4.1. *The Lie algebra \mathfrak{g}_{14} of the group G_{14} is given by*

$$\begin{aligned}\mathfrak{g}_{14} &= \{R \in \mathfrak{e}_8^C \\ &\quad \mid \tilde{\kappa}(\text{ad } R) = (\text{ad } R) \tilde{\kappa}, (\tilde{\mu}_\delta(\text{ad } R)) R' = ((\text{ad } R) \tilde{\mu}_\delta) R', R' \in (V^C)^{14}\} \\ &= \left\{ \left(\Phi \left(D + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & d_1 \\ 0 & -\bar{d}_1 & 0 \end{pmatrix} \right)^\sim + \begin{pmatrix} \tau_1 & 0 & 0 \\ 0 & \tau_2 & t_1 \\ 0 & \bar{t}_1 & \tau_3 \end{pmatrix}^\sim, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha_2 & a_1 \\ 0 & \bar{a}_1 & \alpha_3 \end{pmatrix}, \right. \right. \\ &\quad \left. \begin{pmatrix} 0 & 0 & 0 \\ 0 & \beta_2 & b_1 \\ 0 & \bar{b}_1 & \beta_3 \end{pmatrix}, \nu \right), \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & p_1 \\ 0 & \bar{p}_1 & \rho_3 \end{pmatrix}, \begin{pmatrix} \rho_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \rho \right), \\ &\quad \left(\begin{pmatrix} \zeta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \zeta_2 & z_1 \\ 0 & \bar{z}_1 & \zeta_3 \end{pmatrix}, \zeta, 0 \right), r, 0, 0 \Big) \\ &\quad \left| \begin{array}{l} D \in \mathfrak{so}(8)^C, \tau_i, \alpha_i, \beta_i, \nu, \rho_i, \rho, \zeta_i, \zeta, r \in C, \tau_1 + \tau_2 + \tau_3 = 0, \\ d_1, t_1, a_1, b_1, p_1, z_1 \in \mathfrak{C}, \tau_1 + \frac{2}{3}\nu + 2r = 0 \end{array} \right\}.\end{aligned}$$

In particular, we have

$$\dim_C(\mathfrak{g}_{14}) = 28 + 63 = 91.$$

Proposition 4.2. $G_{14} \cong Spin(14, C)$.

Proof. Let $SO(14, C) = SO((V^{14})^C)$. Then, we have $G_{14}/\mathbf{Z}_2 \cong SO(14, C)$, $\mathbf{Z}_2 = \{1, \sigma\}$. Therefore, G_{14} is isomorphic to $Spin(14, C)$ as a double covering group of $SO(14, C)$. (In detail, see [2]). \square

We define subgroups G_{14}^{com} , G_{13}^{com} and G_{12}^{com} of the group E_8 by

$$G_{14}^{\text{com}} = \{\alpha \in G_{14} \mid \tau\tilde{\lambda}\alpha = \alpha\tau\tilde{\lambda}\},$$

$$G_{13}^{\text{com}} = \{\alpha \in G_{14}^{\text{com}} \mid \alpha(\Phi_1, 0, 0, 0, 1, 0) = (\Phi_1, 0, 0, 0, 1, 0)\},$$

$$G_{12}^{\text{com}} = \{\alpha \in G_{13}^{\text{com}} \mid \alpha(\Phi_1, 0, 0, 0, -1, 0) = (\Phi_1, 0, 0, 0, -1, 0)\},$$

respectively.

Lemma 4.3. $\alpha \in (E_7)^{\kappa, \mu} = Spin(12)$ satisfies

$$\alpha\Phi(0, E_1, 0, 0)\alpha^{-1} = \Phi(0, E_1, 0, 0) \text{ and } \alpha\Phi(0, 0, E_1, 0)\alpha^{-1} = \Phi(0, 0, E_1, 0).$$

Proof. We consider an 11-dimensional sphere $(S')^{11}$ by

$$\begin{aligned} (S')^{11} &= \{P' \in (V')^{12} \mid (P, P)_{\mu} = 1\} \\ &= \{P' = (\xi E_1, \eta E_2 - \tau\eta E_3 + F_1(y), \tau\xi, 0) \\ &\quad \mid \xi, \eta \in C, y \in \mathfrak{C}, (\tau\eta)\eta + \bar{y}y + (\tau\xi)\xi = 1\}. \end{aligned}$$

Since the group $Spin(12)$ acts on $(S')^{11}$, we can put

$$\alpha(E_1, 0, 1, 0) = (\xi E_1, \eta E_2 - \tau\eta E_3 + F_1(y), \tau\xi, 0) \in (S')^{11}.$$

Now, since $1/2\Phi(0, E_1, 0, 0) = (E_1, 0, 1, 0) \times (E_1, 0, 1, 0)$, we have

$$\begin{aligned} 1/2\alpha\Phi(0, E_1, 0, 0)\alpha^{-1} &= \alpha((E_1, 0, 1, 0) \times (E_1, 0, 1, 0))\alpha^{-1} \\ &= \alpha(E_1, 0, 1, 0) \times \alpha(E_1, 0, 1, 0) \\ &= (\xi E_1, \eta E_2 - \tau\eta E_3 + F_1(y), \tau\xi, 0) \times (\xi E_1, \eta E_2 - \tau\eta E_3 + F_1(y), \tau\xi, 0) \\ &= 1/2\Phi(0, ((\tau\eta)\eta + \bar{y}y + (\tau\xi)\xi)E_1, 0, 0). \end{aligned}$$

Since $\alpha(E_1, 0, 1, 0) \in (S')^{11}$, we have $(\tau\eta)\eta + \bar{y}y + (\tau\xi)\xi = 1$. Thus, we obtain $\alpha(E_1, 0, 1, 0) \times \alpha(E_1, 0, 1, 0) = 1/2\Phi(0, E_1, 0, 0)$, that is, $\alpha\Phi(0, E_1, 0, 0)\alpha^{-1} = \Phi(0, E_1, 0, 0)$. Since $\alpha \in Spin(12) \subset E_7$ satisfies $\alpha\tau\lambda = \tau\lambda\alpha$, we have also $\alpha\Phi(0, 0, E_1, 0)\alpha^{-1} = \Phi(0, 0, E_1, 0)$. \square

Proposition 4.4. $G_{12}^{\text{com}} = Spin(12)$.

Proof. Now, let $\alpha \in G_{12}^{\text{com}}$. From

$\alpha(\Phi_1, 0, 0, 0, 1, 0) = (\Phi_1, 0, 0, 0, 1, 0)$, $\alpha(\Phi_1, 0, 0, 0, -1, 0) = (\Phi_1, 0, 0, 0, -1, 0)$, we have $\alpha(0, 0, 0, 0, 1, 0) = (0, 0, 0, 0, 1, 0)$. Hence, since $\alpha \in G_{12}^{\text{com}} \subset E_8$, we see that $\alpha \in E_7$. We first show that $\kappa\alpha = \alpha\kappa$. Since $G_{12}^{\text{com}} \subset E_7$, it

suffices to consider the actions on \mathfrak{P}^C . Since $\alpha \in G_{12}^{\text{com}}$ satisfies $\tilde{\kappa}\alpha = \alpha\tilde{\kappa}$, from

$$\tilde{\kappa}\alpha P = \kappa\alpha P - \alpha P \text{ and } \alpha\tilde{\kappa}P = \alpha\kappa P - \alpha P, \quad P \in \mathfrak{P}^C,$$

we have $\kappa\alpha = \alpha\kappa$. Next, we show that $\mu\alpha = \alpha\mu$. Again, from

$$\alpha(\Phi_1, 0, 0, 0, 1, 0) = (\Phi_1, 0, 0, 0, 1, 0), \quad \alpha(\Phi_1, 0, 0, 0, -1, 0) = (\Phi_1, 0, 0, 0, -1, 0),$$

we have $\alpha(\Phi_1, 0, 0, 0, 0, 0) = (\Phi_1, 0, 0, 0, 0, 0)$. Hence, since $\alpha \in E_7$, we have $\alpha\Phi_1\alpha^{-1} = \Phi_1$, that is, $\alpha\Phi(0, E_1, 0, 0)\alpha^{-1} = \Phi(0, E_1, 0, 0)$. Consequently

$$\begin{aligned} \alpha(\Phi(0, 0, E_1, 0), 0, 0, 0, 0, 1) &= \alpha(-\tilde{\mu}_\delta(\Phi(0, E_1, 0, 0), 0, 0, 0, 1, 0)) \\ &= -\tilde{\mu}_\delta\alpha(\Phi(0, E_1, 0, 0), 0, 0, 0, 1, 0) \\ &= -\tilde{\mu}_\delta(\Phi(0, E_1, 0, 0), 0, 0, 0, 1, 0) \\ &= (\Phi(0, 0, E_1, 0), 0, 0, 0, 0, 1). \end{aligned}$$

Similarly, we have

$$\alpha(\Phi(0, 0, E_1, 0), 0, 0, 0, 0, -1) = (\Phi(0, 0, E_1, 0), 0, 0, 0, 0, -1).$$

Hence, we have

$$\alpha(\Phi(0, 0, E_1, 0), 0, 0, 0, 0, 0) = (\Phi(0, 0, E_1, 0), 0, 0, 0, 0, 0).$$

Moreover, from $\alpha \in E_7$, we have $\alpha\Phi(0, 0, E_1, 0)\alpha^{-1} = \Phi(0, 0, E_1, 0)$. Hence, put together with $\alpha\Phi(0, E_1, 0, 0)\alpha^{-1} = \Phi(0, E_1, 0, 0)$, we have $\alpha\Phi(0, E_1, E_1, 0)\alpha^{-1} = \Phi(0, E_1, E_1, 0)$, that is, $\alpha\mu\alpha^{-1} = \mu$. Thus, we have $\mu\alpha = \alpha\mu$. Therefore, $\alpha \in (E_7)^{\kappa, \mu} = Spin(12)$.

Conversely, let $\alpha \in Spin(12)$. For $R \in \mathfrak{e}_8^C$,

$$\begin{aligned} \tilde{\kappa}\alpha R &= [(\kappa, 0, 0, -1, 0, 0), (\alpha\Phi\alpha^{-1}, \alpha P, \alpha Q, r, u, v)] \\ &= [(\kappa, \alpha\Phi\alpha^{-1}), \kappa\alpha P - \alpha P, \kappa\alpha Q + \alpha Q, 0, -2u, 2v] \end{aligned}$$

and

$$\begin{aligned} \alpha\tilde{\kappa}R &= \alpha[(\kappa, 0, 0, -1, 0, 0), (\Phi, P, Q, r, u, v)] \\ &= [\alpha(\kappa, 0, 0, -1, 0, 0), \alpha(\Phi, P, Q, r, u, v)] \\ &= ([\alpha\kappa\alpha^{-1}, \alpha\Phi\alpha^{-1}], \alpha\kappa\alpha^{-1}(\alpha P) - \alpha P, \alpha\kappa\alpha^{-1}(\alpha Q) + \alpha Q, 0, -2u, 2v). \end{aligned}$$

From $\kappa\alpha = \alpha\kappa$, we have $[\alpha\kappa\alpha^{-1}, \alpha\Phi\alpha^{-1}] = [\kappa, \alpha\Phi\alpha^{-1}]$. Thus, we have $\tilde{\kappa}\alpha R = \alpha\tilde{\kappa}R$, that is, $\tilde{\kappa}\alpha = \alpha\tilde{\kappa}$. Next, from $\mu\alpha = \alpha\mu$ and Lemma 4.3, we have

$$\mu_1(\alpha\Phi_1\alpha^{-1})\mu_1^{-1} = \alpha(\mu_1\Phi_1\mu_1^{-1})\alpha^{-1} = \alpha\Phi(0, 0, E_1, 0)\alpha^{-1} = \Phi(0, 0, E_1, 0).$$

Hence, for $R = (\zeta \Phi_1, P, 0, 0, u, 0) \in (V^C)^{14}$,

$$\begin{aligned}\tilde{\mu}_\delta \alpha R &= \tilde{\mu}_\delta (\zeta \alpha \Phi_1 \alpha^{-1}, \alpha P, 0, 0, u, 0) \\ &= (\Phi(0, 0, -uE_1, 0), 0, i\mu_1 \alpha P, 0, 0, -\zeta)\end{aligned}$$

and

$$\begin{aligned}\alpha \tilde{\mu}_\delta R &= \alpha (\Phi(0, 0, -uE_1, 0), 0, i\mu_1 P, 0, 0, -\zeta) \\ &= (\alpha \Phi(0, 0, -uE_1, 0) \alpha^{-1}, 0, i\alpha \mu_1 P, 0, 0, -\zeta) \\ &= (\Phi(0, 0, -uE_1, 0), 0, i\alpha \mu_1 P, 0, 0, -\zeta).\end{aligned}$$

Hence, from $\mu\alpha = \alpha\mu$, we have $\tilde{\mu}_\delta \alpha R = \alpha \tilde{\mu}_\delta R$, $R \in (V^C)^{14}$. From Lemma 4.3, we have $\alpha(\Phi_1, 0, 0, 0, 0, 0) = (\Phi_1, 0, 0, 0, 0, 0)$. Moreover, since $\alpha \in E_7$, we have

$$\alpha(0, 0, 0, 0, 1, 0) = (0, 0, 0, 0, 1, 0), \quad \alpha(0, 0, 0, 0, -1, 0) = (0, 0, 0, 0, -1, 0).$$

Hence, we have

$$\alpha(\Phi_1, 0, 0, 0, 1, 0) = (\Phi_1, 0, 0, 0, 1, 0), \quad \alpha(\Phi_1, 0, 0, 0, -1, 0) = (\Phi_1, 0, 0, 0, -1, 0).$$

Therefore, $\alpha \in G_{12}^{\text{com}}$. Thus, the proof of the proposition is completed. \square

Lemma 4.5. *The Lie algebras $\mathfrak{g}_{14}^{\text{com}}$ and $\mathfrak{g}_{13}^{\text{com}}$ of the groups G_{14}^{com} and G_{13}^{com} are given respectively by*

$$\begin{aligned}\mathfrak{g}_{14}^{\text{com}} &= \{R \in \mathfrak{g}_{14} \mid \tau \tilde{\lambda}(\text{ad } R) = (\text{ad } R) \tau \tilde{\lambda}\} \\ &= \left\{ \left(\Phi \left(D + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & d_1 \\ 0 & -\bar{d}_1 & 0 \end{pmatrix} + i \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & t_1 \\ 0 & \bar{t}_1 & \epsilon_3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & a_1 \\ 0 & \bar{a}_1 & \rho_3 \end{pmatrix} \right), \right. \\ &\quad \left. - \tau \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & a_1 \\ 0 & \bar{a}_1 & \rho_3 \end{pmatrix}, \nu \right), \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \zeta_2 & z_1 \\ 0 & \bar{z}_1 & \zeta_3 \end{pmatrix}, \begin{pmatrix} \zeta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \zeta \right), \\ &\quad \left. - \tau \lambda \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \zeta_2 & z_1 \\ 0 & \bar{z}_1 & \zeta_3 \end{pmatrix}, \begin{pmatrix} \zeta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \zeta \right), r, 0, 0 \right) \\ &\quad \left| \begin{array}{l} D \in \mathfrak{so}(8), \quad \epsilon_i \in \mathbf{R}, \quad \rho_i, \zeta_i, \zeta \in C, \quad \nu, r \in i\mathbf{R}, \quad \epsilon_1 + \epsilon_2 + \epsilon_3 = 0, \\ i\epsilon_1 + \frac{2}{3}\nu + 2r = 0, \quad d_1, t_1 \in \mathfrak{C}, \quad a_1, z_1 \in \mathfrak{C}^C \end{array} \right\},\end{aligned}$$

$$\begin{aligned} \mathfrak{g}_{13}^{\text{com}} &= \{R \in \mathfrak{g}_{14}^{\text{com}} \mid (\text{ad } R)(\Phi_1, 0, 0, 0, 1, 0) = 0\} \\ &= \left\{ \left(\Phi \left(D + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & d_1 \\ 0 & -\bar{d}_1 & 0 \end{pmatrix} \right) + i \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & t_1 \\ 0 & \bar{t}_1 & \epsilon_3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & a_1 \\ 0 & \bar{a}_1 & \rho_3 \end{pmatrix}, \right. \right. \\ &\quad \left. \left. - \tau \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & a_1 \\ 0 & \bar{a}_1 & \rho_3 \end{pmatrix}, \nu \right), \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \zeta_2 & z_1 \\ 0 & \bar{z}_1 & -\tau\zeta_2 \end{pmatrix}, \begin{pmatrix} \zeta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \tau\zeta_1 \right), \right. \\ &\quad \left. - \tau\lambda \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \zeta_2 & z_1 \\ 0 & \bar{z}_1 & -\tau\zeta_2 \end{pmatrix}, \begin{pmatrix} \zeta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \tau\zeta_1 \right), 0, 0, 0 \right) \\ &\quad \left| \begin{array}{l} D \in \mathfrak{so}(8), \epsilon_i \in \mathbf{R}, \rho_i, \zeta_i \in C, \nu \in i\mathbf{R}, \epsilon_1 + \epsilon_2 + \epsilon_3 = 0, \\ i\epsilon_1 + \frac{2}{3}\nu = 0, d_1, t_1, z_1 \in \mathfrak{C}, a_1 \in \mathfrak{C}^C \end{array} \right\}. \end{aligned}$$

In particular, we have

$$\dim(\mathfrak{g}_{14}^{\text{com}}) = 28 + 63 = 91, \quad \dim(\mathfrak{g}_{13}^{\text{com}}) = 28 + 50 = 78.$$

Lemma 4.6. (1) For $a \in \mathfrak{C}$, we define a C -linear transformation $\epsilon_{13}(a)$ of \mathfrak{e}_8^C by

$$\epsilon_{13}(a) = \exp(\text{ad}(0, (F_1(a), 0, 0, 0), (0, F_1(a), 0, 0), 0, 0, 0)).$$

Then, $\epsilon_{13}(a) \in G_{13}^{\text{com}}$ (Lemma 4.5). The action of $\epsilon_{13}(a)$ on V^{13} is given by

$$\begin{aligned} &\epsilon_{13}(a)(\Phi(0, \zeta E_1, 0, 0), (\xi E_1, \eta E_2 - \tau \eta E_3 + F_1(y), \tau \xi, 0), 0, 0, -\zeta, 0) \\ &= (\Phi(0, \zeta' E_1, 0, 0), (\xi' E_1, \eta' E_2 - \tau \eta' E_3 + F_1(y'), \tau \xi', 0), 0, 0, -\zeta', 0), \\ &\quad \begin{cases} \zeta' &= \zeta \cos|a| - \frac{(a, y)}{2|a|} \sin|a|, \\ \xi' &= \xi, \\ \eta' &= \eta, \\ y' &= y + \frac{2\zeta a}{|a|} \sin|a| - \frac{2(a, y)a}{|a|^2} \sin^2 \frac{|a|}{2}. \end{cases} \end{aligned}$$

(2) For $t \in \mathbf{R}$, we define a C -linear transformation $\theta_{13}(t)$ of \mathfrak{e}_8^C by

$$\theta_{13}(t) = \exp(\text{ad}(0, (0, -tE_1, 0, -t), (tE_1, 0, t, 0), 0, 0, 0)).$$

Then, $\theta_{13}(t) \in G_{13}^{\text{com}}$ (Lemma 4.5). The action of $\theta_{13}(t)$ on V^{13} is given by

$$\begin{aligned} \theta_{13}(t)(\Phi(0, \zeta E_1, 0, 0), (\xi E_1, \eta E_2 - \tau \eta E_3 + F_1(y), \tau \xi, 0), 0, 0, -\zeta, 0) \\ = (\Phi(0, \zeta' E_1, 0, 0), (\xi' E_1, \eta' E_2 - \tau \eta' E_3 + F_1(y'), \tau \xi', 0), 0, 0, -\zeta', 0), \\ \begin{cases} \zeta' &= \zeta \cos t - \frac{1}{4}(\tau \xi + \xi) \sin t, \\ \xi' &= \frac{1}{2}(\xi - \tau \xi) + \frac{1}{2}(\xi + \tau \xi) \cos t + 2\zeta \sin t, \\ \eta' &= \eta, \\ y' &= y. \end{cases} \end{aligned}$$

Lemma 4.7. $G_{13}^{\text{com}}/G_{12}^{\text{com}} \simeq S^{12}$.

In particular, G_{13}^{com} is connected.

Proof. Let $S^{12} = \{R \in V^{13} \mid (R, R)_\mu = 1\}$. The group G_{13}^{com} acts on $(S^C)^{12}$. We shall show that this action is transitive. To prove this, it suffices to show that any $R \in S^{12}$ can be transformed to $1/2(\Phi_1, 0, 0, 0, -1, 0) \in S^{12}$. Now, for a given

$$R = (\Phi(0, \zeta E_1, 0, 0), (\xi E_1, \eta E_2 - \tau \eta E_3 + F_1(y), \tau \xi, 0), 0, 0, -\zeta, 0) \in S^{12},$$

choose $a \in \mathfrak{C}$ such that $|a| = \pi/2$, $(a, y) = 0$. Operate $\epsilon_{13}(a) \in G_{13}^{\text{com}}$ (Lemma 4.6 (1)) on R . Then, we have

$$\epsilon_{13}(a)R = (0, (\xi E_1, \eta E_2 - \tau \eta E_3 + F_1(y'), \tau \xi, 0), 0, 0, 0, 0) = R_1 \in (S')^{11} \subset S^{12},$$

where $(S')^{11} = \{R \in (V')^{12} \mid (R, R)_\mu = 1\}$. Here, since the group $Spin(12)$ ($\subset G_{13}^{\text{com}}$) acts transitively on $S^{11} = \{P \in V^{12} \mid (P, P)_\mu = 1\}$, there exists $\beta \in Spin(12)$ such that $\beta P = (0, E_1, 0, 1)$ for any $P \in S^{11}$. Hence, we have

$$\begin{aligned} \beta R_1 &= \beta(0, P', 0, 0, 0, 0) = (0, \beta P', 0, 0, 0, 0) \\ &= (0, \beta \mu P, 0, 0, 0, 0) = (0, \mu \beta P, 0, 0, 0, 0) \\ &= (0, \mu(0, E_1, 0, 1), 0, 0, 0, 0) = (0, (E_1, 0, 1, 0), 0, 0, 0, 0) \\ &= R_2 \in (S')^{11}, \end{aligned}$$

where $P \in S^{11}$.

Finally, operate $\theta_{13}(-\pi/2) \in G_{13}^{\text{com}}$ (Lemma 4.6 (2)) on R_2 . Then, we have

$$\theta_{13}(-\pi/2)R_2 = \frac{1}{2}(\Phi_1, 0, 0, 0, -1, 0).$$

This shows the transitivity. The isotropy subgroup at $1/2(\Phi_1, 0, 0, 0, -1, 0)$ of G_{13}^{com} is obviously G_{12}^{com} . Thus, we have the homeomorphism

$$G_{13}^{\text{com}}/G_{12}^{\text{com}} \simeq S^{12}. \quad \square$$

Proposition 4.8. $G_{13}^{\text{com}} \cong \text{Spin}(13)$.

Proof. Since the group G_{13}^{com} is connected (Lemma 4.7), we can define a homomorphism $\pi: G_{13}^{\text{com}} \rightarrow SO(13) = SO(V^{13})$ by

$$\pi(\alpha) = \alpha|V^{13}.$$

$\text{Ker } \pi = \{1, \sigma\} = \mathbf{Z}_2$. Since $\dim(\mathfrak{g}_{13}^{\text{com}}) = 78$ (Lemma 4.5) $= \dim(\mathfrak{so}(13))$, π is onto. Hence, $G_{13}^{\text{com}}/\mathbf{Z}_2 \cong SO(13)$. Therefore, G_{13}^{com} is isomorphic to $\text{Spin}(13)$ as a double covering group of $SO(13) = SO(V^{13})$. \square

Proposition 4.9. $G_{14}^{\text{com}} \cong \text{Spin}(14)$.

Proof. Since the group G_{14}^{com} acts on V^{14} and G_{14}^{com} is connected (Proposition 4.2), we can define a homomorphism $\pi: G_{14}^{\text{com}} \rightarrow SO(14) = SO(V^{14})$ by

$$\pi(\alpha) = \alpha|V^{14}.$$

$\text{Ker } \pi = \{1, \sigma\} = \mathbf{Z}_2$. Since $\dim(\mathfrak{g}_{14}^{\text{com}}) = 91$ (Lemma 4.5) $= \dim(\mathfrak{so}(14))$, π is onto. Hence, $G_{14}^{\text{com}}/\mathbf{Z}_2 \cong SO(14)$. Therefore, G_{14}^{com} is isomorphic to $\text{Spin}(14)$ as a double covering group of $SO(14) = SO(V^{14})$. \square

Now, we shall consider the following group

$$\begin{aligned} & ((\text{Spin}(13))^{\sigma'})_{(0, F_1(y), 0, 0)^-} \\ &= \left\{ \alpha \in (\text{Spin}(13))^{\sigma'} \mid \begin{array}{l} \alpha(0, (0, F_1(y), 0, 0), 0, 0, 0, 0) \\ = (0, (0, F_1(y), 0, 0), 0, 0, 0, 0) \end{array} \text{ for all } y \in \mathfrak{C} \right\}. \end{aligned}$$

Lemma 4.10. The Lie algebra $((\mathfrak{spin}(13))^{\sigma'})_{(0, F_1(y), 0, 0)^-}$ of the group $((\text{Spin}(13))^{\sigma'})_{(0, F_1(y), 0, 0)^-}$ is given by

$$\begin{aligned} & ((\mathfrak{spin}(13))^{\sigma'})_{(0, F_1(y), 0, 0)^-} \\ &= \{R \in (\mathfrak{spin}(13))^{\sigma'} \mid (\text{ad } R)(0, (0, F_1(y), 0, 0), 0, 0, 0, 0) = 0\} \\ &= \left\{ \left(\Phi \left(i \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{pmatrix}, -\tau \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{pmatrix}, \nu \right), \right. \\ & \quad \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \zeta_2 & 0 \\ 0 & 0 & -\tau\zeta_2 \end{pmatrix}, \begin{pmatrix} \zeta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \tau\zeta_1 \right), \\ & \quad \left. -\tau\lambda \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \zeta_2 & 0 \\ 0 & 0 & -\tau\zeta_2 \end{pmatrix}, \begin{pmatrix} \zeta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \tau\zeta_1 \right), 0, 0, 0 \right) \\ & \quad \left. \mid \epsilon_i \in \mathbf{R}, \rho_i, \zeta_i \in C, \nu \in i\mathbf{R}, \epsilon_1 + \epsilon_2 + \epsilon_3 = 0, i\epsilon_1 + \frac{2}{3}\nu = 0 \right\}, \end{aligned}$$

In particular, we have

$$\dim(((\mathfrak{spin}(13))^{\sigma'})_{(0,F_1(y),0,0)-}) = 10.$$

Lemma 4.11. $((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)-}/Spin(4) \simeq S^4$.

In particular, $((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)-}$ is connected.

Proof. We define a 5-dimensional \mathbf{R} -vector spaces W^5 by

$$\begin{aligned} W^5 &= \{R \in V^{13} \mid \sigma' R = R\} \\ &= \{R = (\Phi(0, \zeta E_1, 0, 0), (\xi E_1, \eta E_2 - \tau \eta E_3, \tau \xi, 0), 0, 0, -\zeta, 0) \\ &\quad \mid \zeta \in \mathbf{R}, \xi, \eta \in C\} \end{aligned}$$

with the norm

$$(R, R)_\mu = \frac{1}{30} B_8(\tilde{\mu}_\delta R, R) = 4\zeta^2 + (\tau\eta)\eta + (\tau\xi)\xi.$$

Then, $S^4 = \{R \in W^5 \mid (R, R)_\mu = 1\}$ is a 4-dimensional sphere. The group $((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)-}$ acts on S^4 . We shall show that this action is transitive. To prove this, it suffices to show that any $R \in S^4$ can be transformed to $1/2(\Phi_1, 0, 0, 0, -1, 0) \in S^4$ under the action of $((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)-}$. Now, for a given

$$R = (\Phi(0, \zeta E_1, 0, 0), (\xi E_1, \eta E_2 - \tau \eta E_3, \tau \xi, 0), 0, 0, -\zeta, 0) \in S^4,$$

choose $t \in \mathbf{R}$, $0 \leq t < \pi$ such that $\tan t = \frac{4\zeta}{\xi + \tau\xi}$ (if $\xi + \tau\xi = 0$, let $t = \pi/2$).

Operate $\theta_{13}(t) \in ((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)-}$ (Lemmas 4.6 (2), 4.10) on R . Then, we have

$$\theta_{13}(t)R = (0, (\xi' E_1, \eta E_2 - \tau \eta E_3, \tau \xi', 0), 0, 0, 0, 0) = R_1 \in S^3 \subset S^4.$$

Since the group $((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)} \subset ((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)-}$ acts transitively on S^3 (Lemma 3.14), there exists $\beta \in ((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)}$ such that

$$\beta R_1 = (0, (E_1, 0, 1, 0), 0, 0, 0, 0) = R_2 \in S^3.$$

Finally, operate $\theta_{13}(-\pi/2) \in ((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)-}$ on R_2 . Then, we have

$$\theta_{13}(-\pi/2)R_2 = \frac{1}{2}(\Phi_1, 0, 0, 0, -1, 0).$$

This shows the transitivity. The isotropy subgroup at $1/2(\Phi_1, 0, 0, 0, -1, 0)$ of $((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)-}$ is $((Spin(12))^{\sigma'})_{(0,F_1(y),0,0)}$ (Lemma 4.7) = $Spin(4)$. Thus, we have the homeomorphism

$$((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)-}/Spin(4) \simeq S^4. \quad \square$$

Proposition 4.12. $((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)-} \cong Spin(5)$.

Proof. Since $((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)-}$ is connected (Lemma 4.11), we can define a homomorphism $\pi: ((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)-} \rightarrow SO(5) = SO(W^5)$ by

$$\pi(\alpha) = \alpha|W^5.$$

$\text{Ker } \pi = \{1, \sigma\} = \mathbf{Z}_2$. Since $\dim(((\mathfrak{spin}(13))^{\sigma'})_{(0,F_1(y),0,0)-}) = 10$ (Lemma 4.10) $= \dim(\mathfrak{so}(5))$, π is onto. Hence, $((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)-} / \mathbf{Z}_2 \cong SO(5)$. Therefore, $((Spin(13))^{\sigma'})_{(0,F_1(y),0,0)-}$ is isomorphic to $Spin(5)$ as a double covering group of $SO(5)$. \square

Lemma 4.13. *The Lie algebra $(\mathfrak{spin}(13))^{\sigma'}$ of the group $(Spin(13))^{\sigma'}$ is given by*

$$\begin{aligned} & (\mathfrak{spin}(13))^{\sigma'} \\ &= \left\{ \left(\Phi \left(D + i \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{pmatrix}, -\tau \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{pmatrix}, \nu \right), \right. \\ & \quad \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \zeta_2 & 0 \\ 0 & 0 & -\tau\zeta_2 \end{pmatrix}, \begin{pmatrix} \zeta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \tau\zeta_1 \right), \\ & \quad \left. -\tau\lambda \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \zeta_2 & 0 \\ 0 & 0 & -\tau\zeta_2 \end{pmatrix}, \begin{pmatrix} \zeta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \tau\zeta_1 \right), 0, 0, 0 \right) \\ & \quad \left| \begin{array}{l} D \in \mathfrak{so}(8), \epsilon_i \in \mathbf{R}, \rho_i, \zeta_i \in C, \nu \in i\mathbf{R}, \\ \epsilon_1 + \epsilon_2 + \epsilon_3 = 0, i\epsilon_1 + \frac{2}{3}\nu = 0 \end{array} \right\}. \end{aligned}$$

In particular, we have

$$\dim((\mathfrak{spin}(13))^{\sigma'}) = 28 + 10 = 38.$$

Now, we shall determine the group structure of $(Spin(13))^{\sigma'}$.

Theorem 4.14.

$$(Spin(13))^{\sigma'} \cong (Spin(5) \times Spin(8)) / \mathbf{Z}_2, \quad \mathbf{Z}_2 = \{(1, 1), (-1, \sigma)\}.$$

Proof. Let $Spin(13) = G_{13}^{\text{com}}$, $Spin(5) = ((Spin(13))^{\sigma'})_{(0, F_1(y), 0, 0)-}$ and $Spin(8) = ((F_4)_{E_1})^{\sigma'} \subset ((E_6)_{E_1})^{\sigma'} \subset ((E_7)^{\kappa, \mu})^{\sigma'} \subset (G_{13}^{\text{com}})^{\sigma'}$ (Theorem 1.2, Propositions 4.4, 4.8). Now, we define a map $\varphi: Spin(5) \times Spin(8) \rightarrow (Spin(13))^{\sigma'}$ by

$$\varphi(\alpha, \beta) = \alpha\beta.$$

Then, φ is well-defined: $\varphi(\alpha, \beta) \in (Spin(13))^{\sigma'}$. Since $[R_D, R_5] = 0$ for $R_D = (\Phi(D, 0, 0, 0), 0, 0, 0, 0) \in \mathfrak{spin}(8)$, $R_5 \in \mathfrak{spin}(5) = ((\mathfrak{spin}(13))^{\sigma'})_{(0, F_1(y), 0, 0)-}$ (Proposition 4.12), we have $\alpha\beta = \beta\alpha$. Hence, φ is a homomorphism. $\text{Ker } \varphi = \{(1, 1), (-1, \sigma)\} = \mathbf{Z}_2$. Since $(Spin(13))^{\sigma'}$ is connected and $\dim(\mathfrak{spin}(5) \oplus \mathfrak{spin}(8)) = 10$ (Lemma 4.10) $+28 = 38 = \dim((\mathfrak{spin}(13))^{\sigma'})$ (Lemma 4.13), φ is onto. Thus, we have the isomorphism

$$(Spin(5) \times Spin(8))/\mathbf{Z}_2 \cong ((Spin(13))^{\sigma'})^{\sigma'}. \quad \square$$

Now, we shall consider the following group

$$\begin{aligned} & ((Spin(14))^{\sigma'})_{(0, F_1(y), 0, 0)-} \\ &= \left\{ \alpha \in (Spin(14))^{\sigma'} \mid \begin{array}{l} \alpha(0, (0, F_1(y), 0, 0), 0, 0, 0, 0) \\ = (0, (0, F_1(y), 0, 0), 0, 0, 0, 0) \text{ for all } y \in \mathfrak{C} \end{array} \right\}. \end{aligned}$$

Lemma 4.15. *The Lie algebra $((\mathfrak{spin}(14))^{\sigma'})_{(0, F_1(y), 0, 0)-}$ of the group $((Spin(14))^{\sigma'})_{(0, F_1(y), 0, 0)-}$ is given by*

$$\begin{aligned} & ((\mathfrak{spin}(14))^{\sigma'})_{(0, F_1(y), 0, 0)-} \\ &= \{R \in (\mathfrak{spin}(14))^{\sigma'} \mid (\text{ad } R)(0, (0, F_1(y), 0, 0), 0, 0, 0, 0) = 0\} \\ &= \left\{ \left(\Phi \left(i \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix} \right)^{\sim}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{pmatrix}, \right. \right. \\ & \quad \left. \left. - \tau \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{pmatrix}, \nu \right), \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \zeta_2 & 0 \\ 0 & 0 & \zeta_3 \end{pmatrix}, \begin{pmatrix} \zeta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \zeta \right), \right. \\ & \quad \left. - \tau \lambda \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \zeta_2 & 0 \\ 0 & 0 & \zeta_3 \end{pmatrix}, \begin{pmatrix} \zeta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \zeta \right), r, 0, 0 \right) \\ & \quad \left. \left| \begin{array}{l} \epsilon_i \in \mathbf{R}, \rho_i, \zeta_i, \zeta \in C, \nu, r \in i\mathbf{R}, \\ \epsilon_1 + \epsilon_2 + \epsilon_3 = 0, i\epsilon_1 + \frac{2}{3}\nu + 2r = 0 \end{array} \right\} \right\}. \end{aligned}$$

In particular, we have

$$\dim(((\mathfrak{spin}(14))^{\sigma'})_{(0,F_1(y),0,0)^-}) = 15.$$

Lemma 4.16. For $t \in \mathbf{R}$, we define a C -linear transformation $\theta_{14}(t)$ of \mathfrak{e}_8^C by

$$\theta_{14}(t) = \exp(\text{ad}(0, (0, itE_1, 0, it), (itE_1, 0, it, 0), 0, 0, 0)).$$

Then, $\theta_{14}(t) \in ((Spin(14))^{\sigma'})_{(0,F_1(y),0,0)^-}$ (Lemma 4.15). The action of $\theta_{14}(t)$ on V^{14} is given by

$$\begin{aligned} \theta_{14}(t) &(\Phi(0, \zeta E_1, 0, 0), (\xi E_1, \eta E_2 - \tau \eta E_3 + F_1(y), \tau \xi, 0), 0, 0, -\tau \zeta, 0) \\ &= (\Phi(0, \zeta' E_1, 0, 0), (\xi' E_1, \eta' E_2 - \tau \eta' E_3 + F_1(y'), \tau \xi', 0), 0, 0, -\tau \zeta', 0), \\ &\begin{cases} \zeta' &= \frac{1}{2}(\zeta + \tau \zeta) + \frac{1}{2}(\zeta - \tau \zeta) \cos t - \frac{i}{4}(\xi + \tau \xi) \sin t, \\ \xi' &= \frac{1}{2}(\xi - \tau \xi) + \frac{1}{2}(\xi + \tau \xi) \cos t - i(\zeta - \tau \zeta) \sin t, \\ \eta' &= \eta, \\ y' &= y. \end{cases} \end{aligned}$$

Lemma 4.17. $((Spin(14))^{\sigma'})_{(0,F_1(y),0,0)^-} / Spin(5) \simeq S^5$.

In particular, $((Spin(14))^{\sigma'})_{(0,F_1(y),0,0)^-}$ is connected.

Proof. We define a 6-dimensional \mathbf{R} -vector space W^6 by

$$\begin{aligned} W^6 &= \{R \in V^{14} \mid \sigma' R = R\} \\ &= \{R = (\Phi(0, \zeta E_1, 0, 0), (\xi E_1, \eta E_2 - \tau \eta E_3, \tau \xi, 0), 0, 0, -\tau \zeta, 0) \\ &\quad \mid \zeta, \xi, \eta \in C\} \end{aligned}$$

with the norm

$$(R, R)_\mu = \frac{1}{30} B_8(\tilde{\mu}_\delta R, R) = 4(\tau \zeta) \zeta + (\tau \eta) \eta + (\tau \xi) \xi.$$

Then, $S^5 = \{R \in W^6 \mid (R, R)_\mu = 1\}$ is a 5-dimensional sphere. The group $((Spin(14))^{\sigma'})_{(0,F_1(y),0,0)^-}$ acts on S^5 . We shall show that this action is transitive. To prove this, it suffices to show that any $R \in S^5$ can be transformed to $1/2(i\Phi_1, 0, 0, 0, i, 0) \in S^5$ under the action of $((Spin(14))^{\sigma'})_{(0,F_1(y),0,0)^-}$. Now, for a given

$$R = (\Phi(0, \zeta E_1, 0, 0), (\xi E_1, \eta E_2 - \tau \eta E_3, \tau \xi, 0), 0, 0, -\tau \zeta, 0) \in S^5,$$

choose $t \in \mathbf{R}$, $0 \leq t < \pi$ such that $\tan t = -\frac{2i(\zeta - \tau \zeta)}{\xi + \tau \xi}$ (if $\xi + \tau \xi = 0$, let $t = \pi/2$). Operate $\theta_{14}(t) \in ((Spin(14))^{\sigma'})_{(0,F_1(y),0,0)^-}$ (Lemmas 4.15, 4.16)

on R . Then, we have

$$\begin{aligned}\theta_{14}(t)R &= (\Phi(0, (\zeta'E_1, 0, 0), (\xi'E_1, \eta E_2 - \tau\eta E_3, \tau\xi', 0), 0, 0, -\zeta', 0)) \\ &= R_1 \in S^4 \subset S^5.\end{aligned}$$

Since the group $((Spin(13))^{\sigma'})_{(0, F_1(y), 0, 0)^-}$ ($\subset ((Spin(14))^{\sigma'})_{(0, F_1(y), 0, 0)^-}$) acts transitively on S^4 (Lemma 4.11), there exists $\beta \in ((Spin(13))^{\sigma'})_{(0, F_1(y), 0, 0)^-}$ such that

$$\beta R_1 = \frac{1}{2}(\Phi_1, 0, 0, 0, -1, 0) = R_2 \in S^3.$$

Moreover, operate $\theta_{14}(\pi/2)$ and $\alpha(\pi/4)$ (Lemma 3.13) in order,

$$\theta_{14}(\pi/2)R_2 = (0, (-iE_1, 0, i, 0), 0, 0, 0, 0) = R_3,$$

and

$$\alpha(\pi/4)R_3 = (0, (E_1, 0, 1, 0), 0, 0, 0, 0) = R_4.$$

Finally, operate $\theta_{14}(-\pi/2) \in ((Spin(14))^{\sigma'})_{(0, F_1(y), 0, 0)^-}$ on R_4 . Then, we have

$$\theta_{14}(-\pi/2)R_4 = \frac{1}{2}(i\Phi_1, 0, 0, 0, i, 0).$$

This shows the transitivity. The isotropy subgroup at $1/2(i\Phi_1, 0, 0, 0, i, 0)$ of $((Spin(14))^{\sigma'})_{(0, F_1(y), 0, 0)^-}$ is $((Spin(13))^{\sigma'})_{(0, F_1(y), 0, 0)^-}$ (Proposition 4.8) $= Spin(5)$. Thus, we have the homeomorphism

$$((Spin(14))^{\sigma'})_{(0, F_1(y), 0, 0)^-} / Spin(5) \simeq S^5. \quad \square$$

Proposition 4.18. $((Spin(14))^{\sigma'})_{(0, F_1(y), 0, 0)^-} \cong Spin(6)$.

Proof. Since $((Spin(14))^{\sigma'})_{(0, F_1(y), 0, 0)^-}$ is connected (Lemma 4.17), we can define a homomorphism $\pi: ((Spin(14))^{\sigma'})_{(0, F_1(y), 0, 0)^-} \rightarrow SO(6) = SO(W^6)$ by

$$\pi(\alpha) = \alpha|_{W^6}.$$

$\text{Ker } \pi = \{1, \sigma\} = \mathbf{Z}_2$. Since $\dim(((\mathfrak{spin}(14))^{\sigma'})_{(0, F_1(y), 0, 0)^-}) = 15$ (Lemma 4.15) $= \dim(\mathfrak{so}(6))$, π is onto. Hence, $((Spin(14))^{\sigma'})_{(0, F_1(y), 0, 0)^-} / \mathbf{Z}_2 \cong SO(6)$. Therefore, $((Spin(14))^{\sigma'})_{(0, F_1(y), 0, 0)^-}$ is isomorphic to $Spin(5)$ as a double covering group of $SO(6)$. \square

Lemma 4.19. *The Lie algebra $(\mathfrak{spin}(14))^{\sigma'}$ of the group $((Spin(14))^{\sigma'})$ is given by*

$$\begin{aligned} & (\mathfrak{spin}(14))^{\sigma'} \\ &= \left\{ \left(\Phi \left(D + i \begin{pmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{pmatrix}, \right. \right. \\ & \quad - \tau \begin{pmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{pmatrix}, \nu \Big), \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \zeta_2 & 0 \\ 0 & 0 & \zeta_3 \end{pmatrix}, \begin{pmatrix} \zeta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \zeta \right), \\ & \quad \left. - \tau \lambda \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \zeta_2 & 0 \\ 0 & 0 & \zeta_3 \end{pmatrix}, \begin{pmatrix} \zeta_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, 0, \zeta \right), r, 0, 0 \right) \\ & \quad \left| \begin{array}{l} D \in \mathfrak{so}(8), \epsilon_i \in \mathbf{R}, \rho_i, \zeta_i, \zeta \in C, \nu \in i\mathbf{R}, \\ \epsilon_1 + \epsilon_2 + \epsilon_3 = 0, i\epsilon_1 + \frac{2}{3}\nu + 2r = 0 \end{array} \right\}. \end{aligned}$$

In particular, we have

$$\dim((\mathfrak{spin}(14))^{\sigma'}) = 28 + 15 = 43.$$

Now, we shall determine the group structure of $(Spin(14))^{\sigma'}$.

Theorem 4.20.

$$(Spin(14))^{\sigma'} \cong (Spin(6) \times Spin(8))/\mathbf{Z}_2, \quad \mathbf{Z}_2 = \{(1, 1), (-1, \sigma)\}.$$

Proof. Let $Spin(14) = G_{14}^{\text{com}}$, $Spin(6) = ((Spin(14))^{\sigma'})_{(0, F_1(y), 0, 0)-}$ and $Spin(8) = ((F_4)_{E_1})^{\sigma'} \subset ((E_6)_{E_1})^{\sigma'} \subset ((E_7)^{\kappa, \mu})^{\sigma'} \subset (G_{13}^{\text{com}})^{\sigma'} \subset (G_{14}^{\text{com}})^{\sigma'}$ (Theorem 1.2, Propositions 4.8, 4.9). Now, we define a map $\varphi: Spin(6) \times Spin(8) \rightarrow (Spin(14))^{\sigma'}$ by

$$\varphi(\alpha, \beta) = \alpha\beta.$$

Then, φ is well-defined: $\varphi(\alpha, \beta) \in (Spin(14))^{\sigma'}$. Since $[R_D, R_6] = 0$ for $R_D = (\Phi(D, 0, 0, 0), 0, 0, 0, 0) \in \mathfrak{spin}(8)$, $R_6 \in \mathfrak{spin}(6) = ((\mathfrak{spin}(14))^{\sigma'})_{(0, F_1(y), 0, 0)-}$ (Proposition 4.18), we have $\alpha\beta = \beta\alpha$. Hence, φ is a homomorphism. $\text{Ker } \varphi = \{(1, 1), (-1, \sigma)\} = \mathbf{Z}_2$. Since $(Spin(14))^{\sigma'}$ is connected and $\dim(\mathfrak{spin}(6) \oplus \mathfrak{spin}(8)) = 15$ (Lemma 4.15) $+ 28 = 43 = \dim((\mathfrak{spin}(14))^{\sigma'})$ (Lemma 4.19), φ is onto. Thus, we have the isomorphism

$$(Spin(6) \times Spin(8))/\mathbf{Z}_2 \cong ((Spin(14))^{\sigma'}). \quad \square$$

ACKNOWLEDGMENT

The author records here his warmest gratitude to Professor Ichiro Yokota, who recommended to write this paper and has constantly encouraged him.

REFERENCES

- [1] T. IMAI AND I. YOKOTA, *Simply connected compact simple Lie group $E_{8(-248)}$ of type E_8* , J. Math. Kyoto Univ. **21**(1981), 741–762.
- [2] T. MIYASHITA AND I. YOKOTA, *2-graded decompositions of exceptional Lie algebra \mathfrak{g} and group realizations of \mathfrak{g}_{ev} , \mathfrak{g}_0 , Part III, $G = E_8$* , Japanese J. Math. **26**(2000), 31–50.
- [3] T. MIYASHITA AND I. YOKOTA, *Fixed points subgroups $G^{\sigma, \sigma'}$ by two involutive automorphisms σ, σ' of compact exceptional Lie group $G = F_4, E_6$ and E_7* , Math. J. Toyama Univ. **24**(2001), 135–149.
- [4] T. MIYASHITA AND I. YOKOTA, *An explicit isomorphism between $(\mathfrak{e}_8)^{\sigma, \sigma'}$ and $\mathfrak{so}(8) \oplus \mathfrak{so}(8)$ as Lie algebras*, Math. J. Toyama Univ. **24**(2001), 151–158.
- [5] I. YOKOTA, *Realizations of involutive automorphisms σ and G^σ of exceptional linear Lie groups G , Part I, $G = G_2, F_4$ and E_6* , Tsukuba J. Math. **14**(1990), 185–223.
- [6] I. YOKOTA, *Realizations of involutive automorphisms σ and G^σ of exceptional linear Lie groups G , Part II, $G = E_7$* , Tsukuba J. Math. **14**(1990), 379–404.
- [7] I. YOKOTA, *Realizations of involutive automorphisms σ and G^σ of exceptional linear Lie groups G , Part III, $G = E_8$* , Tsukuba J. Math. **14**(1991), 301–314.
- [8] I. YOKOTA, *Exceptional simple Lie groups* (in Japanese), Gendaisuugakusya, Kyoto, 1992.

TOSHIKAZU MIYASHITA
 TOHBU HIGH SCHOOL
 AGATA, TOHBU 389-0517, JAPAN
 e-mail address: serano@janis.or.jp

(Received June 21, 2001)